Université Paris Dauphine
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Introduction to Time series
TD1 - Stationary process

Exercice 1 (Stationarity and strict Stationarity) Suppose X is a random variable with a standard normal distribution $\mathcal{N}(0,1)$, and $Y = X\mathbf{1}_{U=1} - X\mathbf{1}_{U=0}$ where U is a Bernoulli random variable with parameter $1/2$, and independent of X.

1. Show that X and Y have the same distribution;
2. Show that $\text{Cov}(X,Y) = 0$, but X and Y are not independent;
3. Construct a process which is a weak-sense white noise but not a strong-sense white noise.

Exercice 2 (Random walk) Consider a random walk $X = (X_t)_{t\in\mathbb{N}}$ with drift μ : $X_t = \mu + X_{t-1} + Z_t$ for $t \geq 1$ where $X_0 = 0$ and $(Z_t)_{t\in\mathbb{N}}$ is a strong-sense white noise.

1. Calculate the autocovariance function γ_X of X. Is X stationary ?
2. Is the process $(\Delta X_t)_{t\in\mathbb{N}}$ stationary ?

Exercice 3 (Sum of stationary processes) Suppose that $X = (X_t)_{t\in\mathbb{Z}}$ and $Y = (Y_t)_{t\in\mathbb{Z}}$ are two stationary processes and uncorrelated (i.e. $\text{cov}(X_t, Y_s) = 0$ for all s, t). Show that $Z = (Z_t)_{t\in\mathbb{Z}}$ defined by $Z_t = X_t + Y_t$ for all $t \in \mathbb{Z}$ is also stationary. Then find the expression of its autocovariance as a function of those of X and Y.

Exercice 4 (Stationarity of process) Find the stationary processes among the following processes:

1. $X_t = Z_t$ if t is even, $X_t = Z_t + 1$ if t is odd, where $(Z_t)_{t\in\mathbb{Z}}$ is stationary;
2. $X_t = Z_1 + \cdots + Z_t$ where $(Z_t)_{t\in\mathbb{Z}}$ is a strong-sense white noise;
3. $X_t = Z_t + \theta Z_{t-1}$ where $(Z_t)_{t\in\mathbb{Z}}$ is a white noise and $\theta \in \mathbb{R}$ is a constant;
4. $X_t = Z_t Z_{t-1}$ where $(Z_t)_{t\in\mathbb{Z}}$ is a strong-sense white noise;
5. $Y_t = (-1)^t Z_t$ and $X_t = Y_t + Z_t$ where $(Z_t)_{t\in\mathbb{Z}}$ is a strong-sense white noise.

Exercice 5 (Harmonic processes) Consider a process $X = (X_t)_{t\in\mathbb{Z}}$ defined by $X_t = A\cos(\theta t) + B\sin(\theta t)$ for all $t \in \mathbb{Z}$, where A and B are two independent random variables, with zero mean and variance σ^2, and $\theta \in \mathbb{R}$ is a constant.
Is the process X stationary ? Calculate its autocovariance function.

Exercice 6 (Law of large numbers) Consider a stationary process (X_t) with mean μ and the autocovariance γ. Suppose that $\lim_{h\to\infty} \gamma(h) = 0$ (the process is said to be uncorrelated at infinity).
Show that the limit in L^2 of $\frac{1}{T}\sum_{t=1}^{T} X_t$ is equal to μ.

Exercice 7 (Finite difference, trend and moving average) Denote Δ the finite difference operator, which maps a process (X_t) to (ΔX_t) defined by $\Delta X_t = X_t - X_{t-1}$ for all $t \in \mathbb{Z}$. And denote M the moving average operator, which maps a process (X_t) to another process (MX_t) defined by $MX_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1})$ for all $t \in \mathbb{Z}$.

1. If (X_t) is stationary, show that (ΔX_t) and (MX_t) are also stationary and give the expression of their autocovariance as a function of that of (X_t).
2. (X_t) is said to be a process with polynomial trend of degree d ($d \in \mathbb{N}$), if there exists a polynomial P of degree d and a stationary process (U_t) such that $X_t = P(t) + U_t$ for all $t \in \mathbb{Z}$. Show that if (X_t) is a process with polynomial trend of degree d, then (ΔX_t) is a process with polynomial trend of degree $d-1$.
3. Suppose that (X_t) is written as $X_t = P(t) + S(t) + U_t$, where P is a polynomial of degree $d \in \mathbb{N}$, $S : \mathbb{Z} \to \mathbb{R}$ a $R-$periodic function (with $R \in \mathbb{N}^*$) and (U_t) is a stationary process. Give a simple operation transforming (X_t) into a stationary process.
4. What is the effect of M on a process with affine trend (=polynomial trend of degree 1)? And what about the effect of M on a process $X_t = S(t) + U_t$ where $S : \mathbb{Z} \to \mathbb{R}$ is a periodic function with period 3 and (U_t) is stationary ?
5. Give an analogous construction of M which eliminates periodic sequences of degree 3 while keeping polynomials of degree 2 invariant.

Exercice 8 (Property of autocovariance function)

1. Show that the function γ, defined by $\gamma(0) = 1$ and $\gamma(h) = \rho$ for $|h| = 1$ and $\gamma(h) = 0$ otherwise, is an autocovariance function if and only if $|\rho| \leq 1/2$. Give an example of a stationary Gaussian process possessing such an autocovariance function.
2. Same question for γ defined by $\gamma(0) = 1$ and $\gamma(h) = \rho$ if $h \neq 0$, with $0 \leq \rho \leq 1$.
3. Are the following functions autocovariance functions of a stationary process?
   (a) $\gamma(h) = 1$ if $h = 0$ and $\gamma(h) = 1/h$ if $h \neq 0$;
   (b) $\gamma(h) = 1 + \cos(h\pi/2)$;
   (c) $\gamma(h) = (-1)^{|h|}$.